Bayesian Estimation of AR (1) with Change Point under Asymmetric Loss Functions

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Abstract

The object of this paper is a Bayesian analysis of the autoregressive model $X_t = \beta_1 X_{t-1} + \varepsilon_t$,t=1,..,m and $X_t = \beta_2 X_{t-1} + \varepsilon_t$,t=m+1,..,n where $0 < \beta_1, \beta_2 < 1$, and ε_t is independent random variable with an exponential distribution with mean θ_1 but later it was found that there was a change in the process at some point of time m which is reflected in the sequence after ε_m is changed in mean θ_2. The issue this study focused on is at what time and which point the change begins to occur. The estimators of m, β_1, β_2 and θ_1, θ_2 are derived from Asymmetric loss functions namely Linex loss & General Entropy loss functions. Both the non-informative and informative priors are considered. The effects of prior consideration on Bayes estimates of change point are also studied.

Keywords

Bayes Estimates; Change Point; Exponential Distribution; Auto Regressive Process

Introduction

In applications, time series data, the common issue, with characteristics which fail to be compatible with the usual assumption of linearity and Gaussian errors, for a variety of reasons. Bell and Smith (1986) concerned with the autoregressive process AR (1) $X_i = \beta X_{i-1} + \varepsilon_i$ where $0 < \beta < 1$ and ε_i's are i.i.d. and non-negative, who considered the estimation and testing problem for three parametric models: Gaussian, uniform and exponential. For large series, non normality may not be of importance due to the additive nature of filtering processes; but otherwise for small series. Hence, model other than Gaussian, in particular, the exponential, is studied. This AR (1) model can be used as a model for water quality analysis. Let the initial level of pollutant be X_0; and random quantities be $\varepsilon_1, \varepsilon_2, \ldots$ Of that pollutants are "dumped" at regular fixed intervals into the relevant body of water and successive "dumping" a proportion $(1-\beta)$ of the pollutant $(0 \le \beta \le 1)$ is "washed away". If X_i is level of pollutant at time t then $X_i = \beta_i X_{i-1} + \varepsilon_i$ where, ε_i are assumed to have an exponential distribution. Some of the references for Bayesian estimation of parameters of AR (1) as defined above are A. Turkmann, M. A. (1990) and M. Ibazizen, H. Fellage (2003).

Apart from AR (1) with exponential errors, the phenomenon of change point is also observed in several situations in water quality analysis. It may happen that at some point of time instability in the sequence of pollutant level of a river is observed. The issue this study focused on is at what time and which point the change begins to occur, which is called change point inference problem. Bayesian ideas, playing an important role in study of such change point problem has been often proposed as a valid alternative to classical estimation procedure

A sequence of random variables $X_1, \ldots X_m, X_{m+1} \ldots X_n$ is said to have a change point at m $(1 \le m \le n)$ if $X_i \sim F_1(x|\theta_1)$ (i=1,2...m) and $X_i \sim F_2(x|\theta_2)$ (i=m+1,2...n), where $F_1(x|\theta_1) \neq F_2(x|\theta_2)$. The situation is in consideration in which F_1 and F_2 have exponential form, but the change point, m, is unknown. The problem has also been discussed within a Bayesian framework by Chernoff & Zacks (1964), Kander & Zacks (1966), A.F.M. Smith (1975), P.N. Jani & Mayuri Pandya (1999), Pandya, M. and Jani, P.N. (2006), Pandya, M. and Jadav, P. (2009) and Ebrahimi and Ghosh (2001). The Monograph of Broemeling and Tsurumi (1987) on structural changes and a survey by Zacks (1983) are also useful references.

In this paper, an AR (1) model with one change point has been proposed, where the error distribution is supposed to be the changing exponential distribution. In section 2, a change point model related to AR (1) with exponential is developed. In section 3, posterior densities of $\beta_1, \beta_2, \theta_1, \theta_2$ and m for this model are obtained. Bayes estimators of $\beta_1, \beta_2, \theta_1, \theta_2$ and m are

derived from asymmetric loss functions and symmetric loss functions in section 4. A numerical study to illustrate the above technique on generated observations is presented in section 5. In section 6, the sensitivity of the Bayes estimators of m when prior specifications deviate from the true values is studied. In this study, section 7 is about the simulation study implemented by the generation of 10,000 different random samples. The ending of the paper is about the detailed conclusion made on the study.

Proposed AR (1) Model

Let $\{ \varepsilon_n, \ 0 \le \varepsilon_n, \ n \ge 1\}$ be a random sequence having exponential distribution viz.

The p. d. f. of the distribution is given by,

$$f(\varepsilon_i | \theta_1) = 1/\theta_1 \,.\, e^{-\varepsilon_i/\theta_1}, \qquad i = 1,2,3, \ldots . m$$
$$= 1/\theta_2 \ \ e^{-\varepsilon_i/\theta_2}, \qquad i = m+1, \ldots . n$$

Further, let $\{X_n, X_n > 0, n \ge 1\}$ be a sequence of random variables defined as,

$$X_i = \begin{cases} \beta_1 X_{i-1} + \varepsilon_i, & i = 1,2,3 \ldots . m \\ \beta_2 X_{i-1} + \varepsilon_i, & i = m+1, \ldots . n \end{cases} \tag{2.1}$$

With X_0 fixed constant and, $0 < \beta_1, \beta_2 < 1$.

In literature, there are two models for x_0

Model A: x_0 is constant, $x_0=0$ in particular;

Model B: x_0 has the same distribution as that of ε_t.

Here, model A for its flexibility is examined, moreover, the likelihood function (conditional to x_0) for model B is exactly of the same form as that for model A. In the both cases, estimation will be same.

(2.1) is the first order auto regressive process, AR (1), and m is unknown change point in the sequence to be estimated. For $\beta_1 = \beta_2 = \beta$, m = n, the model (2.1) reduces to the model studied by Bell and Smith (1986).

Since, the AR (1) defined in (2.1) is Markov process, the joint p. d. f. of $X_0, X_1, \ldots, X_{m,\ldots}, X_n$ is given by

$$f(X_0, X_1, \ldots, X_n) = \left(\frac{1}{\theta_1}\right)^m . e^{(-s_m + \beta_1 s_m^*)/\theta_1} \left(\frac{1}{\theta_2}\right)^{n-m} e^{\beta_2 (s_n^* - s_m^*)/\theta_2} . e^{-(s_n - s_m)/\theta_2}$$

$$S_k = \sum_{i=1}^{k} X_i \ ; \ S_k^* = \sum_{i=1}^{k} X_{i-1}$$

Since $\varepsilon_i \ge 0$, i = 1, 2, …, n. We have

$X_i \ge \beta_1 . X_{i-1}$, and $X_i > 0$, i = 1, 2, …, m, $X_i \ge \beta_2 . X_{i-1}$, $X_i > 0$, i = m+1, …, n. and as a result $0 < \beta_1, \beta_2 < 1$.

Then, the likelihood function giving the sample information is obtained

X=$(X_1, X_2 \ldots, X_m, X_{m+1}, \ldots X_n)$ is,

$$L(\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X}) = \frac{1}{\theta_1^m} . e^{-A/\theta_1} . (\theta_2)^{-(n-m)} . e^{-B/\theta_2}$$

$$\theta_1, \theta_2 > 0, \ X_0 = 0$$

Where,

A = $S_m - \beta_1 S_m^*$

B=$S_n - S_m - \beta_2 (S_n^* - S_m^*)$ (2.2)

Bayes Estimation

The ML methods as well as other classical approaches are based only on the empirical information provided by the data. However, when there is some technical knowledge on the parameters of the distribution available, a Bayes procedure seems to an attractive inferential method. The Bayes procedure is based on a posterior density, say, $g(\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X})$, which is proportional to the product of the likelihood function $L(\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X})$, with a prior joint density, say, $g(\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X})$ representing uncertainty on the parameters values.

$$g(\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X}) = \frac{L(\beta_1,\beta_2,\theta_1,\theta_2,m \mid \underline{X}) . g(\beta_1,\beta_2,\theta_1,\theta_2,m \mid \underline{X})}{\sum_{m=1}^{n-1} \int_{\beta_1} \int_{\beta_2} \int_{\theta_1} \int_{\theta_2} L(\beta_1,\beta_2,\theta_1,\theta_2,m \mid \underline{X}) . g(\beta_1,\beta_2,\theta_1,\theta_2,m \mid \underline{X}) d\beta_1 d\beta_2 d\theta_1 d\theta_2}$$

Using Informative Priors for β1 and β2, θ1 and θ2

The first step in a Bayes analysis is the choice of the prior density on the distribution parameters. When technical information about the mechanism of process under consideration is accessible, then this information should be converted into a degree of belief on the distribution parameters (or function thereof). This problem is greatly aggravated when some 'physical' meaning can be attached to these parameters.

It is also supposed that some information on these parameters is available, and that this technical knowledge can be given in terms of prior mean values μ_1, μ_2. Independent beta priors on β_1 and $\beta 2$ with respective means $\mu 1$, $\mu 2$ and common standard deviation σ viz are supposed.

$$g(\beta_1) = [\ \beta_1^{a_1 - 1} (1 - \beta_1)^{b_1 - 1}] \mid \beta(a_1, b_1), \qquad 0 < \beta_1 < 1, a_1 > 0, b_1 > 0$$

$g(\beta_2) = [\ \beta_2^{a_2-1}(1-\beta_2)^{b_2-1}]\ \mid \beta\ (a_2, b_2),\quad 0 < \beta_2 < 1, a_2 > 0, b_2 > 0$

If the prior information is given in terms of prior means μ_1, μ_2 and σ, then the hyper parameters can be obtained by solving

$a_i = \sigma_i^{-1}\ [(1-\mu_i)\mu_i^2 - \ \mu_i\sigma_i]$ i = 1,2.

$b_i = \mu_i^{-1}\ (1-\mu_i)a_i$ i = 1,2. (3.1)

Let the joint prior density of θ_1 and θ_2 be

$$g(\theta_1, \theta_2) \propto \frac{1}{\theta_1\theta_2}$$

As in Broemeling et al.(1987), the marginal prior distribution of m is supposed to be discrete uniform over the set { 1, 2, …..n – 1} and ndependent of $\beta 1$, $\beta 2$ and θ_1, θ_2.

$$g(m) = \frac{1}{n-1}$$

For simplicity, it is assumed that, a priori, $\beta_1, \beta_2, \theta_1, \theta_2$ and m are independently distributed. The joint prior density of $\beta_1, \beta_2, \theta_1, \theta_2$ and m say, $g_1(\beta_1, \beta_2, \theta_1, \theta_2, m)$ is,

$$g_1(\beta_1, \beta_2, \theta_1, \theta_2, m) = k_1 \frac{\beta_1^{a_1-1}(1-\beta_1)^{b_1-1} . \beta_2^{a_2-1}(1-\beta_2)^{b_2-1}}{(\theta_1\theta_2)}$$

Where, $k_1 = \frac{1}{\beta(a_1 b_1)\ \beta\ (a_2 b_2)(n-1)}$.

Joint posterior density of $\beta_1, \beta_2, \theta_1, \theta_2$ and m say, $g_1(\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{x})$ is,

$$g_1\ (\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X}) = k_1\, \theta_1^{-(m+1)} e^{-A/\theta_1}\, \theta_2^{-(n-m)-1}\ e^{-B/\theta_2}$$

$$\beta_1^{a_1-1}(1-\ \beta_1)^{b_1-1}\ \beta_2^{a_2-1}(1-\ \beta_2)^{b_2-1} / h_1(\underline{X})\ ,$$

where, $h_1(\underline{X})$ is the marginal density of $\underline{X}$ given as,

$h_1(\underline{X}) = k_1 \sum_{m=1}^{n-1} T_1\ (m)$, (3.2)

$$T_1(m) = = \Gamma m\ \ \Gamma(n-\ m).\ \left\{(\tfrac{1}{a_1})(S_m\ )^{-m}\ \ Appel\ {}_1F_1[\ a_1, m, -\ b_1\ ,\ \ 1+\ a_1, \tfrac{S_m^*}{S_m}\ ,1]\ \right\}$$

$$\left\{(\tfrac{1}{a_2})\ \ [(S_n-\ S_m)]^{-(n-m)}\ \ Appel_1F_1[\ a_2, , n-m, -\ b_2, 1+a_2, \tfrac{S_n^*-S_m^*}{S_n-S_m}, 1]\right\}\ , \tag{3.3}$$

And, $Appel\ {}_1F_1(a, b_1, b_2, c, x, y)$ is Appel Hypergeometric function defined as

Note-1

$$Appel\ {}_1F_1(a, b_1, b_2, c, x, y) = \frac{\Gamma c}{\Gamma a\ \Gamma(c-a)} \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-ux)^{-b_1}(1-uy)^{-b_2}du,$$

Real[a] > 0, Real (c-a) > 0.

Integrating $g_1\ (\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X})$ with (β_1, β_2) and (θ_1, θ_2), leads to the posterior distribution of change point m.

Marginal posterior density of change point m is given by $g_1\ (\mathrm{m} \mid \underline{X}) = \frac{T_1(m)}{\sum_{m=1}^{n-1} T_1(m)}$ (3.4)

$T_1(m)$ is as given in (3.3).

Marginal posterior densities of β_1 and β_2 say, $g_1(\beta_1 \mid \underline{X})$ and $g_2(\beta_2 \mid \underline{X})$ are as,

$$g_1(\beta_1\ \mid \mathrm{X}) = \sum_{m=1}^{n-1} \int_0^1 \int_0^\infty \int_0^\infty g_1\ (\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X})\ .\, d\theta_1 d\theta_2 d\beta_2$$

$$= k_1 \sum_{m=1}^{n-1} \beta_1^{a_1-1}(1-\ \beta_1)^{b_1-1}\ \frac{\Gamma m \Gamma\,(n-m)}{A^m}\left\{\left(\frac{1}{a_2}\right)\ \ [(S_n-\ S_m)]^{-(n-m)}\right.$$

$$\left. .\, Appel\ {}_1F_1[\ a_2, , n-m, -\ b_2, 1+a_2, \frac{S_n^*-S_m^*}{S_n-S_m}, 1]\right\} /\ h_1(\underline{X}) \tag{3.5}$$

$$g_1\ (\beta_2\ \mid \underline{X})\ = k_1 \sum_{m=1}^{n-1} \beta_2^{a_2-1}(1-\ \beta_2)^{b_2-1}\ \frac{\Gamma m \Gamma\,(n-m)}{B^{n-m}}\left\{\left(\frac{1}{a_1}\right)(S_m\ )^{-m}\right.$$

$$\left. Appel\ {}_1F_1[\ a_1, m, -\ b_1\ ,\ \ 1+\ a_1, \frac{S_m^*}{S_m}\ ,1\ ]\right\} /\ h_1(\underline{X}) \tag{3.6}$$

Marginal posterior densities of and θ_2, say $g_1\ (\theta_1\ \mid \underline{X}\ )$ and $g_1\ (\theta_2\ \mid \underline{X}\ )$ are given as,

$$g_1\ (\theta_1\ \mid \underline{X}\ ) = \sum_{m=1}^{n-1} \int_0^1 \int_0^1 \int_0^\infty g_1\ (\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X}) d\theta_2 d\beta_1\, d\beta_2$$

$$=k_1 \sum_{m=1}^{n-1} \theta_1^{-(m+1)} \int_0^1 \frac{\beta_1^{a_1-1}(1-\beta_1)^{b_1-1}}{e^{-A/\theta_1}} d\beta_1 \Gamma(n-m)\{ \left(\frac{1}{a_2}\right) [(S_n - S_m)]^{-(n-m)}$$

$$Appel\ {}_1F_1[\, a_2,, n-m, -b_2, 1+a_2, \frac{S_n^*-S_m^*}{S_n-S_m}, 1\,]/\ h_1(\underline{X}) \tag{3.7}$$

And

$$g_1\,(\theta_2 \mid \underline{X}\,) = \sum_{m=1}^{n-1} \int_0^1 \int_0^1 \int_0^\infty g_1\,(\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X}) d\theta_1 d\beta_1\, d\beta_2$$

$$= k_1 \sum_{m=1}^{n-1} \theta_2^{-(n-m)} \Gamma m \int_0^1 \frac{\beta_2^{a_2-1}(1-\beta_2)^{b_2-1}}{e^{-B/\theta_2}} d\beta_2 \left\{ \left(\frac{1}{a_1}\right)(S_m)^{-m}\right.$$

$$\left. Appel\ {}_1F_1[\, a_1, m, -b_1,\ 1+a_1, \frac{S_m^*}{S_m}, 1]\right\}/\ h_1(\underline{X}) \tag{3.8}$$

Using Non Informative Priors for β_1, β_2, θ_1 and θ_2

The non-informative prior is a density which adds no information to that contained in the empirical data. If there is no available information on $\beta_1, \beta_2, \theta_1, \theta_2$, m which are assumed to be a priori independent random variables, then the non-informative prior is,

$$g_2\,(\beta_1, \beta_2, \theta_1, \theta_2, m\,) \propto \frac{1}{\theta_1 \theta_2\,(n-1)}$$

Joint posterior density of θ_1, θ_2, β_1, β_2 and m say, $g_2\,(\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X})$ is given by

$$= \frac{1}{n-1} \theta_1^{-(m+1)}\, e^{-S_m + \beta_1 S_m^*/\theta_1}\, \theta_2^{-(n-m+1)}\, e^{-S_n + S_m + \beta_2 (S_n^* - S_m^*)/\theta_2} / h_2(\underline{X})$$

Where, $h_2(\underline{x})$ is the marginal density of $\underline{x}$ given by,

$$h_2(\underline{x}) = \frac{1}{(n-1)} \sum_{m=1}^{n-1} T_2\,(m) \tag{3.9}$$

Where,

$$T_2(m) = \Gamma m \Gamma(n-m) \left[-\frac{S_m^{-m} + (S_m - S_m^*)^{-m}}{(S_m^*)(m)} \left(-\frac{(S_n - S_m)^{-(n-m)} + (S_n - S_m - S_n^* + S_m^*)^{-(n-m)}}{(n-m)(S_n^* - S_m^*)} \right) \right] \tag{3.10}$$

Marginal posterior density of change point m is given by

$$g_2(m \mid \underline{X}) = \sum_{m=1}^{n-1} \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 g_2\,(\beta_1, \beta_2, \theta_1, \theta_2, m \mid \underline{X})\, d\theta_1 d\theta_2 d\beta_1 d\beta_2$$

$$= T_2(m) / \sum_{m=1}^{n-1} T_2\,(m) \tag{3.11}$$

Marginal posterior density of θ_1 and θ_2 are given as,

$$g_2(\theta_1 \mid \underline{X}) = \frac{1}{n-1} \sum_{m=1}^{n-1} \theta_1^{-(m+1)} \frac{\theta_1 e^{-S_m/\theta_1}}{S_m^*} \cdot \left(e^{S_m^*/\theta_1} - 1 \right)$$

$$\left(\frac{(S_n - S_m)^{-(n-m)} + (S_n - S_m - S_n^* + S_m^*)^{-(n-m)}}{(n-m)(S_n^* - S_m^*)} \right) /\ h_2(\underline{X}) \tag{3.12}$$

$g_2(\theta_2 \mid \underline{X})$

$$= \frac{1}{n-1} \sum_{m=1}^{n-1} \left[\theta_2^{-(n-m+1)}\, \frac{e^{-\left(\frac{S_n - S_m}{\theta_2}\right)}}{(S_n^* - S_m^*)} \left[e^{\frac{(S_n^* - S_m^*)}{\theta_2}} - 1 \right] \left[-\frac{S_m^{-m} + (S_m - S_m^*)^{-m}}{(S_m^*)(m)} \right] /\ h_2(\underline{X}) \right. \tag{3.13}$$

Marginal posterior density of β_1 and β_2 are given as,

$$g_2(\beta_1 \mid \underline{X}) = \frac{1}{n-1} \sum_{m=1}^{n-1} \Gamma m\, \Gamma(n-m) \frac{1}{(S_m - \beta_1 S_m^*)^m}$$

$$\left[\left(-\frac{(S_n - S_m)^{-(n-m)} + (S_n - S_m - S_n^* + S_m^*)^{-(n-m)}}{(n-m)(S_n^* - S_m^*)} \right) \right] / h_2(\underline{X}) \tag{3.14}$$

$$g_2(\beta_2 \mid \underline{X}) = \frac{1}{n-1} \sum_{m=1}^{n-1} \Gamma m \Gamma(n-m) \frac{1}{(S_n - S_m - \beta_2 (S_n^* - S_m^*))^{(n-m)}}^{-(n-m)}$$

$$[(-\frac{S_m^{-m} + (S_m - S_m^*)^{-m}}{(S_m^*)(m)})]\ / h_2(\underline{X}) \tag{3.15}$$

Remark 1: For n=m, $\beta_1 = \beta_2$, $\theta_1 = \theta_2$ the equations (3.5), (3.7) and (3.12), (3.14) reduce to the marginal posterior densities of β *and* θ of AR(1) process without change point.

Bayes Estimates under Symmetric and Asymmetric Loss Functions

The Bayes estimate of a generic parameter (or function thereof) α based on a Squared Error Loss (SEL) function $L_1 (\alpha, d) = (\alpha-d)^2$, where d is decision rule to estimate α which is the posterior mean. As a consequence, the SEL function is relative to an integer parameter,

$L_1'(m, v) \propto (m-v)^2$, m, v=0,1,2,……

Hence, the Bayesian estimate of an integer-valued parameter under the SEL function $L_1'(m, v)$ is no longer the posterior mean and can be obtained by numerically minimizing the corresponding posterior loss. Generally, such a Bayesian estimate is equal to the nearest integer value to the posterior mean. So, the nearest value to the posterior mean is regarded as Bayes Estimate. The Bayes estimators of m under SEL are the nearest integer values to (4.1) and (4.2),

$$m^* = \sum_{m=1}^{n-1} m\, T_1(m) \ / \sum_{m=1}^{n-1} T_1(m) \qquad (4.1)$$

$$m^{**} = \sum_{m=1}^{n-1} m\, T_2(m) \ / \sum_{m=1}^{n-1} T_2(m) \qquad (4.2)$$

Other Bayes estimators of α based on the loss functions

$L_2 (\alpha, d) = | \alpha - d |$

$$L_3 (\alpha, d) = \begin{cases} 0, & \text{if } |\alpha - d| < \epsilon,\ \epsilon > 0 \\ 1, & \text{otherwise} \end{cases}$$

is the posterior median and posterior mode, respectively.

Asymmetric Loss Function

The Loss function L (α, d) provides a measure of the financial consequences arising from a wrong decision rule d to estimate an unknown quantity α. The choice of the appropriate loss function is just dependent on financial considerations, but independent upon the used estimation procedure. In this section, Bayes estimator of change point m is derived from different asymmetric loss function using both prior considerations explained in section 3.1 and 3.2. A useful asymmetric loss, known as the Linex loss function was introduced by Varian (1975). Under the assumption that the minimal loss at d, the Linex loss function can be expressed as,

$L_4 (\alpha, d) = \exp. [q_1 (d- \alpha)] - q_1 (d - \alpha) - I,\ q_1 \neq 0.$

The sign of the shape parameter q_1 reflects the deviation of the asymmetry, $q_1 > 0$ if over estimation is more serious than under estimation, and vice-versa, and the magnitude of q_1 reflects the degree of asymmetry.

Minimizing expected loss function $E_m [L_4 (m, d)]$ and using posterior distribution (3.4) and (3.11), the Bayes estimate of m using Linex loss function is obtained by means of the nearest integer value to (4.3) under informative and non-informative prior , say, say m_L^* m_L^{**} .

$$m_L^* = -{}^{1}/_{q_1}\ I_n[\textstyle\sum_{m=1}^{n-1} e^{-a_1 m} T_1(m) / \sum_{m=1}^{n-1} T_1(m)],$$

$$m_L^{**} = -{}^{1}/_{q_1}\ I_n[\textstyle\sum_{m=1}^{n-1} e^{-a_1 m} T_2(m) / \sum_{m=1}^{n-1} T_2(m)], \qquad (4.3)$$

Where T_1 (m) and $T_2(m)$ are as given in (3.3) and (3.10).

Minimizing expected loss function $E_m [L_4 (\beta_i, d)]$ and using posterior distributions (3.5) as well as (3.6), the Bayes estimators of β_1 and β_2 are obtained using Linex loss function as,

$$\beta_{1L}^* = -\frac{1}{q_1} \ln \left[E\, e^{-q_1 \beta_1} \right]$$

$$= -\frac{1}{q_1} \ln \Big[\sum_{m=1}^{n-1} \int_0^1 \frac{\beta_1^{a_1-1} (1-\beta_1)^{b_1-1}\, e^{-q_1 \beta_1}}{A^m}\, d\beta_1\, \Gamma m \Gamma(n-m) \Big\{ \left(\frac{1}{a_2}\right) [(S_n - S_m)]^{-(n-m)}$$

$$Appel\ {}_1F_1\Big[a_2,, n-m, -b_2, 1+a_2, \frac{S_n^* - S_m^*}{S_n - S_m}, 1 \Big] \Big\} / h_1(\underline{X}) \qquad (4.4)$$

$$\beta_{2L}^* == -\frac{1}{q_1} \ln \Big[\sum_{m=1}^{n-1} \int_0^1 \frac{\beta_2^{a_2-1} (1-\beta_2)^{b_2-1} . e^{-q_1 \beta_2}}{B^{n-m+1}}\, d\beta_2\, \Gamma m \Gamma(n-m) \{ (S_m)^{-m}$$

$$Appel\ {}_1F_1\Big[a_1, m, -b_1,\ 1+a_1, \frac{S_m^*}{S_m}, 1 \Big] \Big\} / h_1(\underline{X}) \qquad (4.5)$$

Where $Appel\ {}_1F_1[a_1, m, -b_1,\ 1+a_1, \frac{S_m^*}{S_m}, 1]\}$ and $Appel\ {}_1F_1[a_2,, n-m, -b_2, 1+a_2, \frac{S_n^*-S_m^*}{S_n-S_m}, 1]$ are same as in note 1 and $h_1(\underline{X})$ is as in (3.2).

Another loss function, called General Entropy loss function (GEL), proposed by Calabria and Pulcini (1996) is given by,

$L_5(\alpha, d) = (d/\alpha)^{q_3} - q_3\, l_n(d/\alpha) - 1,$

Whose minimum occurs at d = α, minimizing expectation E [L_5 (m, d)] and using posterior density $g_i(m \mid \underline{x})$, i = 1,2. The Bayes estimate of m is achieved using General Entropy loss function by means of the nearest integer value to (4.6) and (4.7) under informative and non-informative prior, say m_E^* and m_E^{**}

$$m_E^* = \left[E_1[m^{-q_3}]\right]^{-1/q_3} = [\textstyle\sum_{m=1}^{n-1} m^{-q_3} T_1(m) / \sum_{m=1}^{n-1} T_1(m)]^{-1/q_3}, \tag{4.6}$$

$$m_E^{**} = \left[E_1[m^{-q_3}]\right]^{-1/q_3} = [\textstyle\sum_{m=1}^{n-1} m^{-q_3} T_2(m) / \sum_{m=1}^{n-1} T_2(m)]^{-1/q_3}, \tag{4.7}$$

Putting q_3 = - 1 in (4.6) and (4.7), Bayes estimate, posterior mean of m is acquired.

Minimizing expected loss function E [L_5 (β_i, d)] and using posterior distributions (3.5) and (3.6), we get the Bayes estimates of β_i using General Entropy loss function respectively as,

$$\beta_{1E}^* = k_1\{\textstyle\sum_{m=1}^{n-1} \Gamma m\ \Gamma(n-m) \frac{1}{a_1-q_3} (S_m)^{-m} \frac{1}{(S_n^*-S_m^*)^{a2}} \frac{1}{a_2} [(S_n - S_m)]^{-(n-m)}$$

$$Appel\ {}_1F_1\left[a_1-q_{3,}\, m, -b_1,\ 1+a_1-q_{3,},\frac{S_m^*}{S_m}, 1\right]$$

$$Appel\ {}_1F_1[a_{2,}, n-m, -b_2, 1+a_2, \frac{S_n^*-S_m^*}{S_n-S_m}, 1]\Big\}\ / h_1(\underline{X})\Big\}^{-1/q_3} \tag{4.8}$$

$$\beta_{2E}^* = k_1\{\textstyle\sum_{m=1}^{n-1} \Gamma m\ \Gamma(n-m) \frac{1}{a_2-q_3} [(S_n - S_m)]^{-(n-m)} \frac{1}{a_1}\{(S_m)^{-m}$$

$$Appel\ {}_1F_1[a_2-q_{3,}, n-m, -b_2, 1+a_2-q_3, \frac{S_n^*-S_m^*}{S_n-S_m}, 1]\}$$

$$Appel\ {}_1F_1\left[a_1, m, -b_1,\ 1+a_1, \frac{S_m^*}{S_m}, 1\right]\Big\}/h_1(\underline{X})\Big\}^{-1/q_3} \tag{4.9}$$

Where $Appel\ {}_1F_1[a_1, m, -b_1,\ 1+a_1, \frac{S_m^*}{S_m}, 1]\}$ and $Appel\ {}_1F_1[a_2, n-m, -b_{2,}, 1+a_{2,}\frac{S_n^*-S_m^*}{S_n-S_m}, 1]$ are same as in note 1 and $h_1(\underline{X})$ is as in (3.2).

Minimizing expected loss function E [L_5 (β_i, d)] and using posterior distributions (3.14) and (3.15), we get the Bayes estimate β_{iE}^{**} of β_i, i = 1, 2 using non informative priors, under General Entropy Loss function is given as,

$$\beta_{1E}^{**} = \Big\{\frac{1}{n-1}\textstyle\sum_{m=1}^{n-1} \Gamma m\ \left[\left(-\frac{(S_n-S_m)^{-(n-m)} + (S_n-S_m-S_n^*+S_m^*)^{-(n-m)}}{(n-m)(S_n^*-S_m^*)}\right)\right]$$

$$(S_m)^{-m}\ {}_2F_1[1-q_3, m, 2-q_3, \frac{S_m^*}{S_m}](1-q_3)^{-1}/\ h_2(\underline{x})\Big\}^{-1/q_3} \tag{4.10}$$

$$\beta_{2E}^{**} = \Big\{\frac{1}{n-1}\sum_{m=1}^{n-1} \Gamma(n-m). - \frac{S_m^{-m} + (S_m - S_m^*)^{-m}}{(S_m^*)(m)}$$

$$(S_n - S_m)^{-(n-m+1)}\ {}_2F_1[1-q_3, n-m, 2-q_3, \frac{S_n^*-S_m^*}{S_n-S_m}](1-q_3)^{-1}/\ h_2(\underline{x})\Big\}^{-1/q_3} \tag{4.11}$$

Where $h_2(\underline{X})$ is same as in (3.9).

Minimizing expected loss function E [L_5 (θ_1, d)] and using posterior distributions (3.7) as well as (3. 12), we get the Bayes estimate θ_{1E}^* and θ_{1E}^{**} of θ_1 using General Entropy Loss function as,

$$\theta_{1E}^* = k_1 \sum_{m=1}^{n-1} \Gamma(m+q_3)\ \Gamma(n-m)\ \frac{1}{a_1}\frac{1}{a_2}(S_m)^{-(m+q_3)}[(S_n - S_m)]^{-(n-m)}$$

$$Appel\ {}_1F_1\left[a_1, m, -b_1,\ 1+a_1, \frac{S_m^*}{S_m}, 1\right]\Big\}\quad Appel\ {}_1F_1[a_2, n-m, -b_{2,}, 1+a_{2,}\frac{S_n^*-S_m^*}{S_n-S_m}, 1\Big\}/h_1(\underline{X})\Big\}^{-\frac{1}{q_3}} \tag{4.12}$$

$$\theta_{1E}^{**} = \Big\{\frac{1}{n-1}\textstyle\sum_{m=1}^{n-1} \Gamma(n-m)\frac{\Gamma(-1+m+q_3)}{S_m^*}\quad \left[-\frac{(S_n-S_m)^{-(n-m)} + (S_n-S_m-S_n^*+S_m^*)^{-(n-m)}}{(n-m)(S_n^*-S_m^*)}\right]$$

$$\left[-\left(\frac{1}{s_m}\right)^{-1+m+q_3} + \left(\frac{1}{s_m-S_m^*}\right)^{-1+m+q_3}\right]/h_2(\underline{X})\Big\}^{-1/q_3} \tag{4.13}$$

$$s_m - s_m^* > 0, s_m > 0, m + q_3 > 0,$$

Minimizing expected loss function E [L_5 (θ_2, d)] and using posterior distributions (3.8) and (3. 13), we get the Bayes estimate θ_{2E}^{*} and θ_{2E}^{**} of θ_2 using General Entropy Loss function as,

$$\theta_{2E}^{*} = \{k_1 \sum_{m=1}^{n-1} \Gamma m \Gamma(n - m + q_3)(\tfrac{1}{a_2}) [(S_n - S_m)]^{-(n-m+q_3)} \left(\tfrac{1}{a_1}\right) (S_m)^{-(m+1)}$$

$$Appel\ {}_1F_1\left[a_2, n-m+q_3, -b_2,, 1+a_2, \frac{S_n^* - S_m^*}{S_n - S_m}, 1\right]$$

$$Appel\ {}_1F_1[a_1, m, -b_1,\ 1+a_1, \tfrac{S_m^*}{S_m}, 1]\Big\}/h_1(\underline{X})\Big\}^{-1/q_3} \tag{4.14}$$

$$\theta_{2E}^{**} = \left\{\frac{1}{n-1}\sum_{m=1}^{n-1} \Gamma m \frac{\Gamma(-1+n-m+q_3)}{(S_n^* - S_m^*)}\right.$$

$$\left[-\left(\frac{1}{S_n - s_m}\right)^{-1+n-m+q_3} + \left(\frac{1}{S_n - s_m - (S_n^* - S_m^*)}\right)^{-1+n-m+q_3}\right]$$

$$\left.\left[-\frac{S_m^{-m} + (S_m^* - S_m^*)^{-m}}{(S_m^*)(m)}\right]/h_2(\underline{X})\right\}^{-1/q_3}$$

$$s_m - s_m^* > 0, s_m > 0, m + q_3 > 0, \tag{4.15}$$

Remark 2: For n=m, $\beta_1 = \beta_2$, $\theta_1 = \theta_2$ the equations (4.4), (4.8), (4.10) and (4.12), (4.13) reduce the Bayes estimates under asymmetric loss function of β *and* θ of AR(1) process without change point under asymmetric loss functions.

Numerical Study

Let us consider AR (1) model as

$$X_1 = \begin{cases} 0.1X_{i-1} + \epsilon_i\ , & i = 1, 2, \dots 12 \\ 0.8\, X_{i-1} + \epsilon_i\ , & i = 13, 14, \dots 20. \end{cases}$$

Where, ϵ_i's are independently distributed exponential distributions given in (2.1) with

θ_1=0.6, θ_2=1. 20 random observations have been generated from proposed AR (1) model given in (2.2). The first twelve observations are from exponential distribution with θ_1=0.6 and next eight are from exponential distribution with θ_2=1. β_1 and β_2 themselves are random observations from beta distributions with prior means μ_1 = 0.1, μ_2 = 0.8 and common standard deviation σ = 0.1 respectively, resulting in a_1 = 0.001,

b =0.09, a_2=0.48 and b_2= 0.12. These observations are given in Table-1.

Posterior mean of *m*, *θ1*, *θ2*, *β1*, and *β2* and the posterior median of *m* have been calculated.. Posterior mode appears to be a bad estimator of m. For a comparative purpose point of view, estimators under the non informative prior are also calculated. The results are shown in Table-2. The Bayes estimates m_L^*, m_E^* *of* m, $\theta_{1E}^*, \theta_{2E}^*$ *of* θ_1 *and* θ_2, β_{1E}^* , β_{2E}^* of *β1* and *β2* respectively for the data given in Table-1 have been computed. As well as the Bayes estimates under the non informative prior and asymmetric loss functions which are also calculated, from which the result shown in Table-3 reveal that m^*_L, m^*E, β_{1E}^*, β_{2E}^*, θ_{1E}^* and θ_{2E}^* are robust with respect to the change in the shape parameter of GE loss function.

TABLE 1 GENERATED OBSERVATIONS FROM PROPOSED AR (1) MODEL.

I	1	2	3	4	5	6	7	8	9	10
X_i	0.91	1.08	1.18	0.26	2.08	0.25	0.17	0.88	1.21	0.44
I	1	2	3	4	5	6	7	8	9	10
ϵ_i	0.90	0.99	1.07	0.15	2.05	0.04	0.14	0.86	1.12	0.32
I	11	12	13	14	15	16	17	18	19	20
X_i	0.38	0.41	0.86	1.68	3.01	2.53	4.36	5.53	4.57	4.61
I	11	12	13	14	15	16	17	18	19	20
ϵ_i	0.33	0.43	0.48	0.99	1.67	0.12	2.33	2.04	0.15	0.95

TABLE 2 THE VALUES OF BAYES ESTIMATES OF CHANGE POINT

Prior Density	Bayes estimates of change point		Bayes estimates of auto correlation coefficient.		Bayes estimates of Exponential parameters	
	Posterior Median	Posterior Mean	β_1	β_2	θ_1	θ_2
Informative	12.02	12	0.1	0.8	0.6	0.95
Non informative	12.22	11	0.12	0.82	0.63	1.03

TABLE 3 THE BAYES ESTIMATES USING ASYMMETRIC LOSS FUNCTIONS.

Prior Density	Shape parameter		Bayes estimates of change point		Bayes estimates under General Entropy Loss			
	q_1	q_3	m_L^*	m_E^*	β_{2E}^*	β_{1E}^*	θ_{1E}^*	θ_{2E}^*
Informative	0.8	0.8	12	12	0.83	0.13	0.13	0.95
	1.2	1.2	12	12	0.82	0.12	0.12	0.94
	1.5	1.5	12	12	0.81	0.11	0.11	0.91
Non informative			m_L^{**}	m_E^{**}	β_{2E}^{**}	β_{1E}^{**}	θ_{1E}^{**}	θ_{2E}^{**}
	0.8	0.8	13	13	0.86	0.16	0.68	1.4
	1.2	1.2	13	13	0.85	0.15	0.66	1.3
	1.5	1.5	13	13	0.83	0.13	0.65	1.2

TABLE-4 POSTERIOR MEAN M* FOR THE DATA GIVEN IN TABLE-1.

μ_1	μ_2	m^*	m^*_E
0.1	0.6	12	12
0.07	0.8	12	12
0.2	0.4	12	12

Sensitivity of Bayes Estimates

In this section, the studied sensitivity of the Bayes estimates is obtained from section 4 with respect to change in the prior of the parameter. The means μ_1, μ_2 and standard deviation σ of beta prior have been used as prior information in computing the parameters a_1, b_1, a_2, b_2 of the prior. Following Calabria and Pulcini (1996), it is also assumed that the prior information should be correct if the true value of β_1 (β_2) is close to prior mean μ_1(μ_2) and is assumed to be wrong if β_1 (β_2) is far from μ_1(μ_2). We have computed m^* and m_E^* using (4.1) and (4.6) for the data given in Table-1 with common value of σ =0.1. For q3 =0.9, considering different values of (μ_1, μ_2) and result are shown in Table-4.

The results shown in Table-4 lead to conclusion that m^* and m^*_E are robust with respect to the correct choice of the prior density of β_1 (β_2) and a wrong choice of the prior density of β_1 (β_2). Moreover, they are also robust with respect to the change in the shape parameter of GE loss function.

Simulation Study

In section 4, we have obtained Bayes estimates of m on the basis of the generated data given in Table-1 for given values of parameters. To justify the results, we have generated 10,000 different random samples with m=12, n=20, θ_1=1.0, θ_2=2.0, β_1= 0.4, β_2 = 0.6 and obtained the frequency distributions of posterior mean, median of m, m_L^*, m_E^* with the correct prior consideration. The results are shown in Table 5. We also obtain the frequency distributions of Bayes estimates of autoregressive coefficients given in section 4 with the both prior considerations and the results are shown in Table 6 and table 7. The value of shape parameter of the general entropy loss and Linex loss is taken as 0.1.

We also simulate several samples from AR (1) model explained section 2 with m=12, n=20, θ_1 =1.0, 0.6, 0.7; θ_2 =2.0, 0.8, 0.9 and β_1=0.2, 0.4, 0.5; β_2=0.4, 0.6, 0.7. For each θ_1, θ_2, β_1 and β_2, and Bayes estimators of change point m and autoregressive coefficients β_1 and β_2 using q_3 = 0.9 has been computed for different prior means μ_1 and μ_2, Which lead to the same conclusion as for single sample, that the Bayes estimators posterior mean of m, and m^*_E are robust with respect to the correct choice of the prior specifications on β_1 (β_2) and wrong choice of the prior specifications on β_2 (β_1)

TABLE 5: FREQUENCY DISTRIBUTIONS OF THE BAYES ESTIMATES OF THE CHANGE POINT

Bayes estimate	% Frequency for		
	01-10	11-13	14-20
Posterior mean	18	70	12
Posterior median	10	74	16
Posterior mode	12	73	15
m_L^*	13	77	10
m_E^*	14	78	8

TABLE 6: FREQUENCY DISTRIBUTIONS OF THE BAYES ESTIMATES OF AUTOREGRESSIVE COEFFICIENTS B1 AND B2 USING GENERAL ENTROPY LOSS FUNCTION (INFORMATIVE)

Bayes estimate	% Frequency for			
	0.1-0.3	0.3-0.5	0.5-0.7	0.7-0.9
β_{1E}^*	13	82	02	03
β_{2E}^*	11	01	84	04

TABLE 7: FREQUENCY DISTRIBUTIONS OF THE BAYES ESTIMATES OF (NON-INFORMATIVE)

Bayes estimate	% Frequency for			
	0.1-0.3	0.3-0.5	0.5-0.7	0.7-0.9
β_{1E}^{**}	13	78	06	03
β_{2E}^{**}	11	01	74	14

Conclusions

Our numerical study shows that the Bayes estimators posterior mean of m, and m^*_E are robust with respect to the different choice of the prior specifications on β_1, β_2 and are sensitive in case prior specifications on both β_1, β_2 deviate simultaneously from the true values. Numerical study also shows that posterior mean of m is sensitive when prior specifications on both β_1, β_2 deviate simultaneously from the true values.

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